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Taking the labor out of Libor.

Zürich, May 20th 2008.

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Investment Solutions

Our Goal Today

- Discuss the rationale behind the Libor Market Model (LMM).

- Discuss the application to Risk Management.

- I am purposely leaving all the mathematics and/or technical details (well almost all… ☺).

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Outline

1. Setting the stage.
2. What's in a Curve?
3. Introduction to market quoted Interest Rate Derivatives
   a. Caps/Floors
   b. Swaptions
4. Libor Market Model.
5. Risk Management.

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1. Setting the stage

Let us assume we have a portfolio that includes Interest Rate dependent products. An example could be a bond whose floating-leg coupon is fixed semiannually as

$$C_i = Max\,(Min\,(15 \times (SR_{10y} - SR_{2y}), L_{6m}), 2\%)$$

We want to compute Value at Risk for our portfolio, how can we do that?

This product depends not only on the general interest rate level, but also on the slope and correlations (and requires convexity correction).

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1. Setting the stage (contd.).

For each single Bond we have several possibilities:

- Look up in Bloomberg.
- Semi-analytic Formulae .

(good for easy cases, not this one, although it could provide an approximation)

- Trees
  - Black-Derman-Toy.
  - Black-Karasinski.
  - Hull-White.
  - Cox-Ingersoll-Ross.
  - Vasicek.
- Heath-Jarrow-Morton.

These could work, but in them one models one (short) rate or even two rates. They are flexible in terms of payoff and early exercise, but too rigid in terms of correlations.

- Libor/Swap Market Model.

Since we model directly Libor Rates, it is more flexible and market oriented. However it is a bit more complicated to implement.

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1. Setting the stage (contd.).

Also, in a portfolio context, models used for different bonds have to be consistent with each other.

So, it seems that we have little choice but to use the Libor Market Model…

To understand the rationale behind it, let us briefly discuss alternative descriptions of interest rate curves and market quoted options.

BTW, it is currently the standard in many banks.

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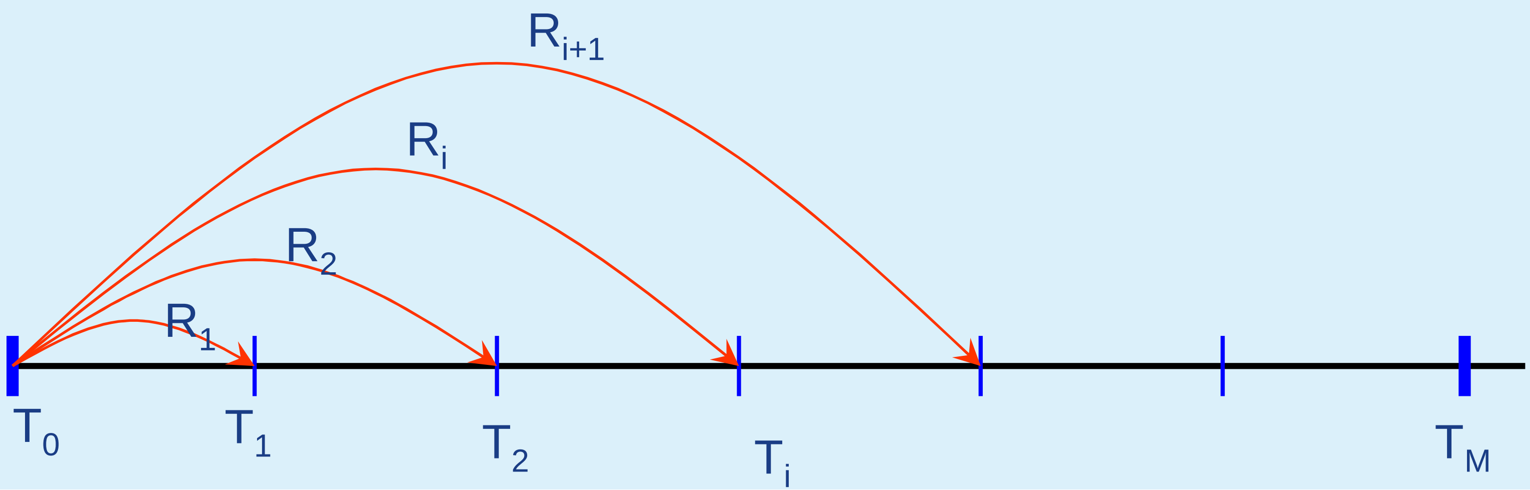
2. What is in a curve?

This looks like a trivial question, but how do we describe an interest rate curve?

- The straightforward way is to use Quoted Rates (T,R).
- Problem: Each rate may actually represent a different "product"
  - Example: Depo vs. Swap (different market conventions)



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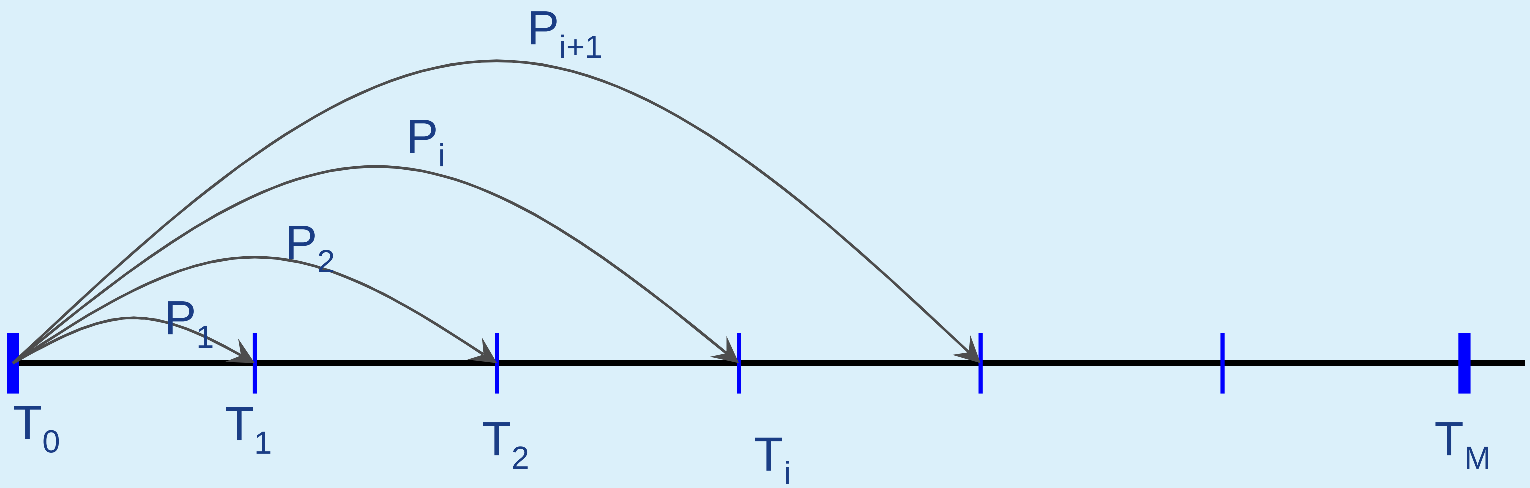
2. What is in a curve? (contd.)

— Zero Bond/Discount Factors (T,P).



1. Advantages
2. No difference in market conventions.
3. A zero bond is "tradable"

$$P_i \equiv P(T_i) = \frac{1}{1 + R_i.(T_i - T_0)}$$

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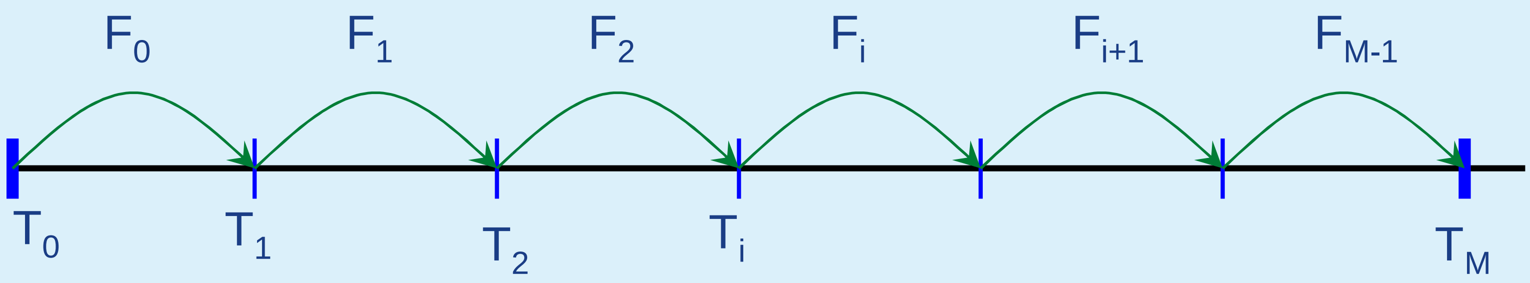
2. What is in a curve? (contd.)

– Forward rates (T, F).



1. Advantages
2. No difference in market conventions.
3. Almost "tradable"
4. Correspondence with option markets.

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2. What is in a curve? (contd.)

These three representations are completely equivalent and one can always map one to the other, for example

$$F_i \equiv \frac{1}{(T_{i+1} - T_i)} \left(\frac{P(T_{i+1})}{P(T_i)} - 1 \right)$$

$$P(T_M) \equiv \prod_{j=0}^{M-1} \frac{1}{1 + F_j \delta_j}$$

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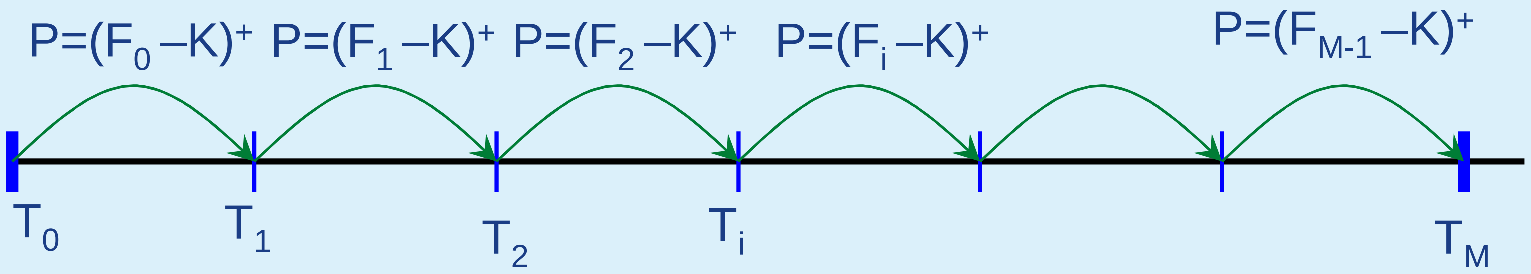
3. Introduction to market quoted IRDs

Caps/Floors

A Cap/Floor is a series of call (put) options on a given Libor rate. Using our previously defined forward rate, a Cap of strike K would as payoff P



$$Cap(K,T_0,T_M) = \sum_{i=0}^{M-1} Max(F_i - K,0)\delta_i P(T_{i+1})$$

$$\delta_i \equiv T_{i+1} - T_i$$

Note: From here on I assume a Nominal N=1

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3. Introduction to market quoted IRDs (contd.)

- We see that a cap is actually a collection of "caplets". Each caplet is a Black-like option with the corresponding Libor rate as underlying.

- The price can be then computed as

$$Cap(K,T_M) = \sum_{i=0}^{M-1} BS(F_i, K, T_i, \sigma_{cap})\delta_i$$

where BS() stands for Black Formula and the Volatility (quoted in the market) is assumed the same for all caplets. This expression assumes each rate is lognormal. Correlations between caplets do not play any role.

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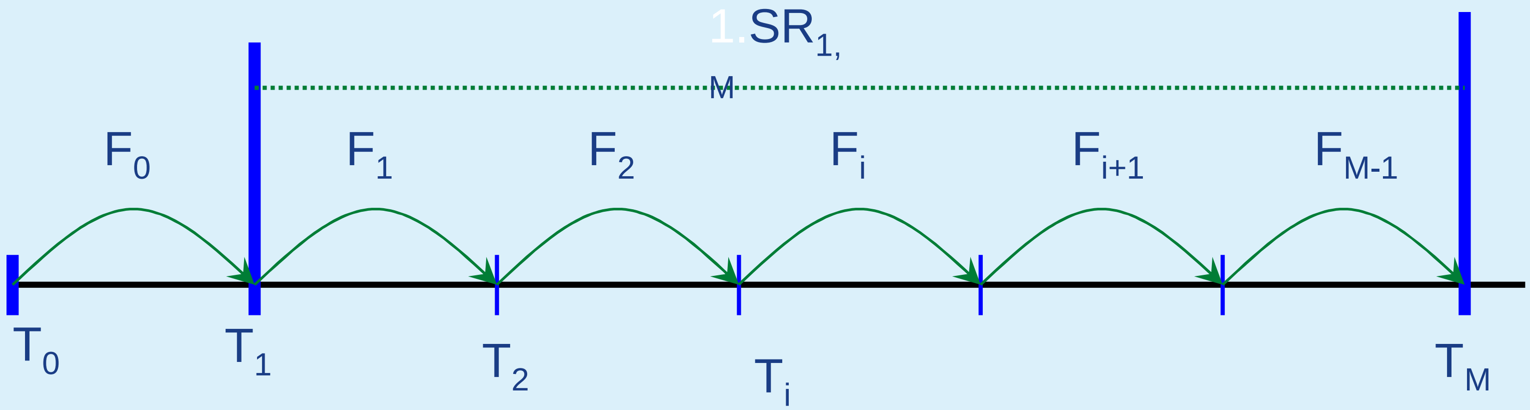
3. Introduction to market quoted IRDs (contd.)

Swaptions

A payer (receiver) Swaption is a call (put) option on a given swap rate.

To fix ideas



$$SR_{1,M} = \frac{\sum_{i=1}^{M-1} F_i \delta_i P(T_{i+1})}{\sum_{i=1}^{M-1} \delta_i P(T_{i+1})} = \frac{P(T_1) - P(T_M)}{\sum_{i=1}^{M-1} \delta_i P(T_{i+1})} \equiv \sum_{i=1}^{M-1} \omega_i F_i$$

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3. Introduction to market quoted IRDs (contd.)

$$Swaption\,(K, T_1, T_M) = Max\left(\sum_{i=1}^{M-1}(F_i - K)\delta_i P(T_{i+1}), 0\right)$$

$$Swaption(K, T_1, T_M) = Max(SR_{1,M} - K, 0)\sum_{i=1}^{M-1}\delta_i P(T_{i+1})$$

We see that a swaption is a Black-like option with the corresponding

Swap rate as underlying. The price can be then computed as

$$Swaption(K, T_1, T_M) = BS(SR_{1,M}, K, T_1, \sigma_{Swp})\sum_{i=1}^{M-1}\delta_i P(T_{i+1})$$

where as before BS() stands for Black Formula and we have only one Volatility (quoted in the market). This expression assumes that the swap rate is lognormal.

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3. Introduction to market quoted IRDs (contd.)

Correlation between forwards plays a role. The payoff cannot be decomposed in a collection of individual options on each forward "swaplets".

This last point about correlations is very important. To understand it we only need to remember that

$$SR_{1,M} = \frac{\sum_{i=1}^{M-1} F_i \delta_i P(T_{i+1})}{\sum_{i=1}^{M-1} \delta_i P(T_{i+1})} \equiv \sum_{i=1}^{M-1} \omega_i F_i$$

i.e. since the swap rate is a weighted-average of the corresponding forward rates, the volatility of the swap rate will include the correlation between forward rates. To make matters worse, the weights are time (and actually forward) dependent.

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3. Introduction to market quoted IRDs (contd.)

Summarizing so far

For Caps and Swaptions:

- One easily get quotes for the Implied Volatilities (Bloomberg).
- Both products are (very) liquid.
- Both products can be easily written in terms of forwards.
- Problem: In both cases the volatility is computed assuming the underlying rate to be lognormal (incompatibility).

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4. Libor Market Model.

The Libor Market Model uses the (T,F) representation of the interest rate curve. That is, our basic quantities will be the forward (Libor) rates. The idea is quite simple.

1. Find Stochastic Differential Equation (SDE) for forward rates.
2. Solve the SDE.
3. Compute the evolution of all forward rates (a.k.a. "The Curve").
4. Given "The Curve", all products that can be written as a finite combination of forward rates can be priced .

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2. Libor Market Model (contd.)

The Stochastic Differential Equation.

It can be shown that

Measure dependent. We ignore that from here on.

$$\frac{dF_i}{F_i} = \mu_i^{(M)}(\mathbf{F},t)dt + \sigma_i(t)dW_t \qquad (1)$$

with

$$\mu_j^{(M)}(\mathbf{F},t) = -\sigma_j(t)\sum_{i=j+1}^{M}\frac{F_i(t)\delta_i}{1+F_i(t)\delta_i}\sigma_i(t)\rho_{j,i}(t) \qquad (2)$$

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4. Libor Market Model (contd.)

Let us not celebrate just yet. There are technical issues that need to be dealt with:

- Need to find one volatility per Forward
  - easy ➔ Bootstrapping.
- Caplets and "swaplets" have different tenor
  - easy ➔ calculate weights differently.
- Drift depends on other forwards
  - medium ➔ needs approximation to avoid prohibitively "long" calculations (we use iterated predictor-corrector)

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4. Libor Market Model (contd.)

- Instantaneous Volatility needs parametrization
  - easy ➔ see Rebonato et al. .
- Instantaneous Correlation needs parametrization.
  - medium ➔ see Rebonato et al. .
- The model requires CALIBRATION.
  - For volatilities easy ➔ almost direct.
  - For correlations complicated ➔ Swaptions are weakly dependent ➔ Correlations may be difficult to calibrate ☹ .

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4. Libor Market Model (contd.)

- No analytical solution
  - medium ➔ Montecarlo Pricing.
    - Requires Variance Reduction Techniques.
    - Greeks require extra work.
    - Some products may be trickier

      Early Termination can be problematic.

      Products requiring continuous monitoring

      Triggered/callable swaptions

      Barrier style.

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5. Risk Management

At this point the question is how do we calculate VaR for our original portfolio?

We simulate the forward rates, construct the curves and revalue our products consistently.

Notice that no use was made of Duration or Convexity or for that matter any other typical notions in Fixed income products. Indeed both parameters can be calculated using its definitions for the portfolio as a whole.

Indeed, we can compute not only VaR but also other risk measures such as CVaR, or even sensitivity to each point of the curve.

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Contact Details

José Luis Vega
Investment Solutions
Risk Management & Analytics
+41-44-234 5755

UBS AG
www.ubs.com

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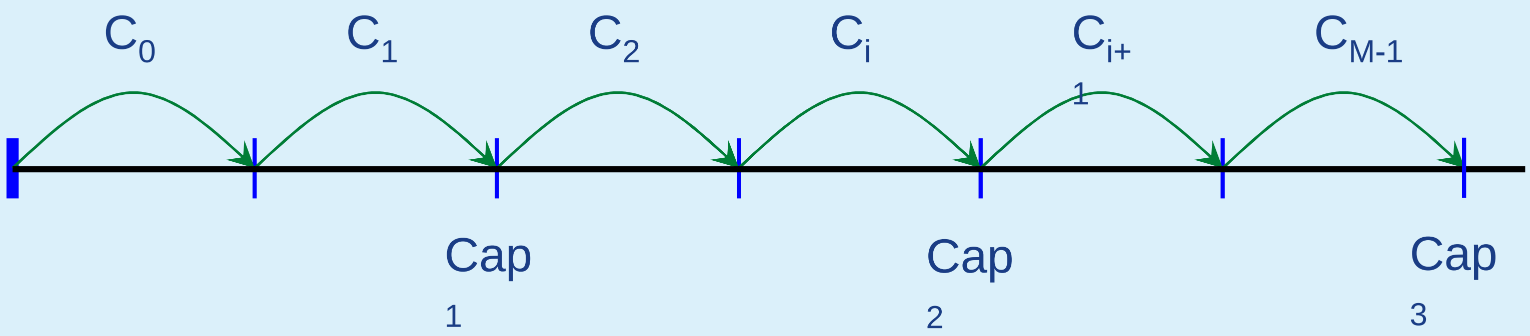
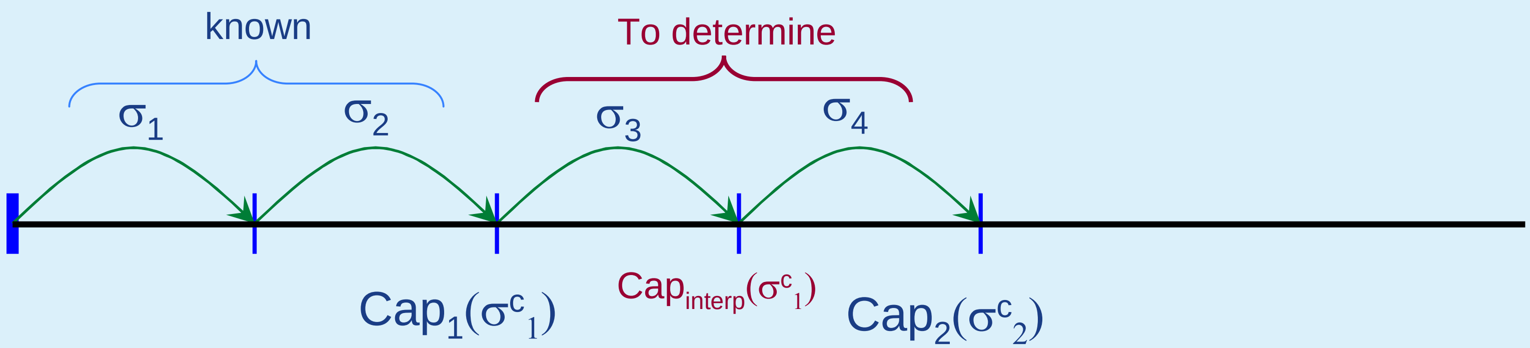
How to compute Caplet Volatilities?

This apparentely harmless question presents us with the first problem



Each new cap brings several new caplets. We need to find a way of, with only one equation, solving for several unknowns .

Appendix A: Caplet volatilities (contd.)



$$Cap(K,T_M) = \sum_{i=0}^{M-1} BS(F_i,K,T_i,\sigma_{cap})\delta_i = \sum_{i=0}^{M-1} BS(F_i,K,T_i,\sigma_i)\delta_i$$

Three Possibilities to close the equation:

1. Assume all new volatilities equal. $\sigma_3 = \sigma_4$

2. Interpolate Caps to get a one-to-one correspondence.

3. Assume all new caplets are interpolated between the last known and the last to find $\sigma_3 = f(\sigma_1, \sigma_2, \sigma_4)$

Appendix B: Swaption Volatilities

You must remember this ...
A swap rate is just a rate ...

$$SR_{1,M} = \sum_{i=1}^{M-1} \omega_i(t) F_i$$

After some calculations we find that

$$\sigma^2_{SR_{p,q}} = \frac{\sum_{i,j=p}^{q-1} \omega_i \omega_j F_i F_j \sigma_i \sigma_j \rho_{i,j}}{SR^2_{p,q}}$$

We have used as approximation (freezing) $\omega_i(t) \cong \omega_i(0)$

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Appendix C: Caplet vs. Swaplet tenor

Caplets and "swaplets" have different tenors

Then instead of

$$SR_{1,M} = \frac{\sum_{i=1}^{M-1} F_i \delta_i P(T_{i+1})}{\sum_{i=1}^{M-1} \delta_i P(T_{i+1})} \equiv \sum_{i=1}^{M-1} \omega_i F_i$$

we use

$$SR_{1,M} = \frac{\sum_{i=1}^{M-1} F_i \delta_i P(T_{i+1})}{\sum_{i=1}^{K-1} \tilde{\delta}_i P(\tilde{T}_{i+1})} \equiv \sum_{i=1}^{M-1} \tilde{\omega}_i F_i$$

and everything returns to normal ....

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Appendix D: Freezing the dynamics

Since the drift depends on the forward(s), we should take very small time-steps in our simulations.

To avoid prohibitively "long" calculations we use predictor-corrector.

This method amounts to freezing the drift term, i.e. if the instantaneous drift is

$$\mu_j(\mathbf{F},t) = -\sigma_j(t)\sum_{i=j+1}^{M}\frac{F_i(t)\delta_i}{1+F_i(t)\delta_i}\sigma_i(t)\rho_{j,i}(t)$$

we define

$$C(s,t) = \int_s^t \sigma_j(u)\sigma_i(u)\rho_{j,i}(u)du$$

to get

$$\mu_j(s,t) = \mu_j(\mathbf{F}(s),C(s,t)) = -\sum_{i=j+1}^{M}\frac{F_i(s)\delta_i}{1+F_i(s)\delta_i}C(s,t)$$

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Appendix D: Freezing the dynamics (contd.)

By freezing the drift we get the Euler solution

$$F_i^E(t) = F_i(s)e^{\mu_i(\mathbf{F}(s),C(s,t)) - \frac{1}{2}C_{i,i}(s,t) + \mathbf{AZ}}$$

where Z is a vector of gaussian distributed random shocks and A is given by

$$A(s,t)A(s,t)^T = C(s,t)$$

we can combine these two drifts to get an effective drift

$$\tilde{\mu}_i(s,t) = \frac{1}{2}\left(\mu_i\left(\mathbf{F}(s),C(s,t)\right) + \mu_i\left(\mathbf{F}^E(t),C(s,t)\right)\right)$$

Finally our forward Libor rate dynamics is given by

$$F_i(t) = F_i(s)e^{\tilde{\mu}_i(s,t) - \frac{1}{2}C_{i,i}(s,t) + \mathbf{AZ}}$$

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Appendix E: Instantaneous Volatility parametrization

We use two different parametrizations

$$\sigma_i(t) = k_i\left(d + \left(a + b\left(T_i - t\right)\right)e^{-c(T_i - t)}\right)$$

Piecewise Constant

$$\sigma_i(t) = k_i \begin{cases} \sigma_1 & t \in [T_0, T_1] \\ \sigma_2 & t \in [T_1, T_2] \\ \sigma_j & t \in [T_{j-1}, T_j] \\ \sigma_K & t \in [T_{K-1}, T_K] \end{cases}$$

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Appendix F: Instantaneous Correlation Parametrization

Rebonato 2

$$\rho_{j,i} = \alpha + (1-\alpha)e^{\beta\left|T_i - T_j\right|}$$

Rebonato 3

$$\rho_{j,i} = \alpha + (1-\alpha)e^{\left(\beta - \eta\left(\max(i,j)-1\right)\right)\left|T_i - T_j\right|}$$

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